Journal of Modern Science and Technology
Vol. 3. No. 1. March 2015 Issue. Pp. 147 –162

Effect on Natural Frequency of a Simply Supported Plate due to Circular Cutouts

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Cutouts are sometimes necessary in plates to fix it or install some machinery on the plate. They can play a vital role by changing the natural frequency of the plate. In this literature, four parameters: position, size, number and orientation of cutouts have been focused to investigate the effect of cutouts on the natural frequency of a simply supported square plate. The results show that cutouts at different positions of the plate change the natural frequencies significantly; also the distribution and orientation of cutouts play an important role to change natural frequencies. Mesmerizing changes in the modal shape of the plate due to cutout is also discussed in this literature. The numerical simulation has been done by FEM. Some results are validated by analytical approaches of published literatures.

Keywords: Simply Supported Plate, Circular Cutout, Cutout Sizes, Cutout Position, Cutout Distribution and Orientation of Cutout, Bare Plate.

Field of Research: Vibration, Mechanical Engineering.

1. Introduction

A square plate with a rectangular or a circular cutout has been widely used as a substructure for ship, aircraft, airplane, machineries, and bridges. Plates with opening due to window, door, and ventilation are often used in both modern and classical aerospace, mechanical and civil engineering. A square plate with circular cutout has special purpose of uses. Plate with cutout may be used to install machineries, to fix the screw, to reduce masses and for different purposes. As the geometrical shape of the plate changes due to cutout, the vibration characteristic of the plate becomes complicated. Plate vibration is one of the major concerns in all industry and excessive vibration may damage the whole system.In this literature we have analyzed the natural frequency of a square plate for different sizes, positions, distributions, orientation and number of cutouts. Natural frequency of the bare plate is compared with the plate with circular cutouts of various conditions. Cutout reduces the total mass of the bare plate and thus changes the natural frequency.

The objective of this literature is to analyze the Natural frequency of a square plate due to circular cutouts on the basis of various constraints whereas all the previous literatures focused on the single or double circular cutouts and the sizes of the cutout. Here Natural frequency of the plate have been analyzed on the basis of (i) changing the percentage of mass reduction of the plate by circular cutout (case 1), and is validated with the ICCM method (Kwak& Han2007) at section 5 of this literature (ii) changing the position of the single circular cutout in the plate (case 2), (iii) distributing the same amount of cutout into number of portions on the plate (case 3 and 4).(iv) changing the orientation of the distributed cutouts (case 5).

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2. Literature Review

Incredible amount of research has been carried out on the free vibration of plates under various boundary conditions and various shapes of cutouts by different analytical methods. KwakandHeo (2007) applied the Independent Coordinate Coupling Method (ICCM) for a rectangular plate with a circular cutout and verified the results with finite element method .Paramasivam(1973) used the finite difference method for plate with a rectangular hole by the classical Rayleigh-Ritz method. Takahashi (1958) used the classical Rayleigh-Ritz method after deriving the total energy by subtracting the energy of the hole from the energy of the whole plate .Lee and Kim (1984) carried out vibration experiments on the rectangular plates with a hole in air and water .Pickett (1961) analyzed the vibration behavior of plates with holes .Aksu and Ali(1976) determined the dynamic characteristic of plates with cutouts using finite difference method. Avalos and Laura(2003) calculated the natural frequency of a simply-supported rectangular plate with two rectangular holes by using the classical Rayleigh-Ritz method. Sakiyama(2003) analyzed the natural vibration characteristics of an orthotropic plate with a square hole by means of the Green function assuming the hole as an extremely thin plate. Lee .(2007) used an indirect formulation in conjunction with degenerate kernels and Fourier series to solve for the natural frequencies and modes of circular plates with multiple circular holes and verified the finite element solution by using ABAQUS.Kwakand Han(2007) presented a new method of Independent Coordinate Coupling Method (ICCM) for the free vibration analysis of a rectangular plate with a rectangular or a circular hole. This method utilizes independent coordinates for the global and local domains and the transformation matrix between the local and global coordinates. In the Rayleigh-Ritz method, the effect of the hole can be considered by the subtraction of the energy for the cutout domain in deriving the total energy.

3. Methodology

A simply supported square plate made of mild steel is considered with dimensions of 60cm ×60cm×0.2cm. The geometry of the bare plate structure under this analysis is shown in Figure 1 indicating different positions of the plate.

The material of the plate is chosen to have the mechanical properties defined by Young's modulus 210 GPa, density 7850 Kg/m^3 and Poisson's ratio 0.3. A Finite Element Method by ANSYS APDL has been carried out to investigate the natural frequency of bare plate and plate with different sizes and orientations of circular cutouts. Here SOLID185is used for 3-D modeling which is defined by eight nodes having three degrees of freedom at each node and Mesh-200 is used as a meshing element.

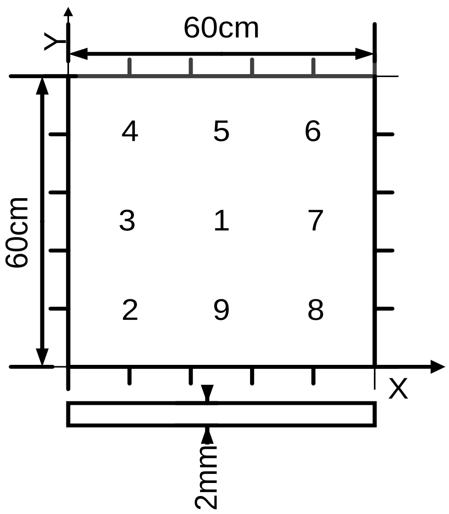


Figure 1: A Simply Supported Square Plate with Uniform Thickness

Case 1:A circular cutout is considered at the middle of the plate and the percentage of mass reduction is changed by changing the size of the cutout (cutout circle radius). The natural frequency is compared with the bare plate of different sizes of cutout. (Figure 2).
Case 2: The single circular cutout of constant radius is placed into several positions on the plate as shown in Figure 3.
Case 3: Same amount of cutout of case 1is equally distributed intotwo cutouts onthe plate with a distance ***d***. Then the natural frequency is compared with the bare plate's natural frequency and with the natural frequency of plate with single cutout (Figure 4).
Case 4: The single circular cutout of case 1 is distributed equally into three, four and five different positions of the plate (Figure 5).
Case 5: The distributed three and five cutouts of case 4 are relocated into different orientation of the plate (Figure 6).

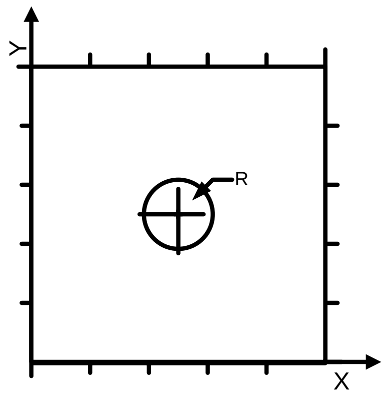
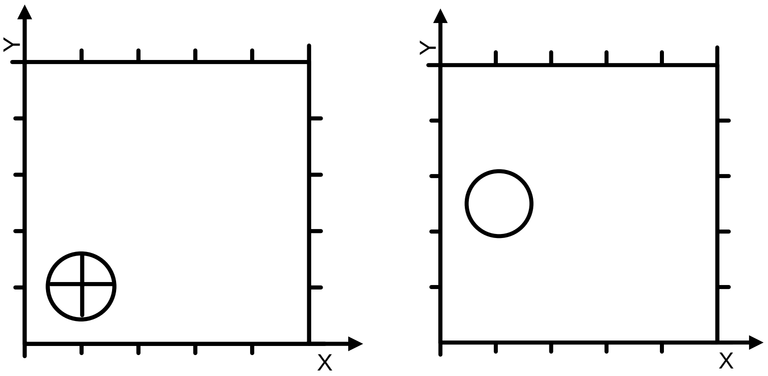


Figure 2: Circular Cutout of Various size (position 1)



position 2

(center at 0.15m, 0.15m)

position 3

(center at 0.15m, 0.3m)

Figure 3: Circular Cutout at Various Positions

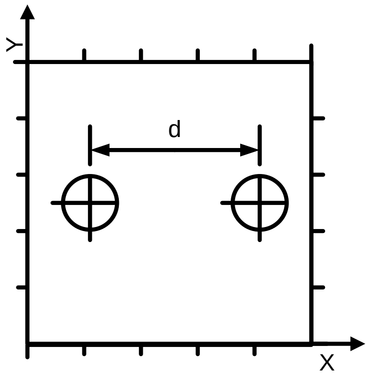
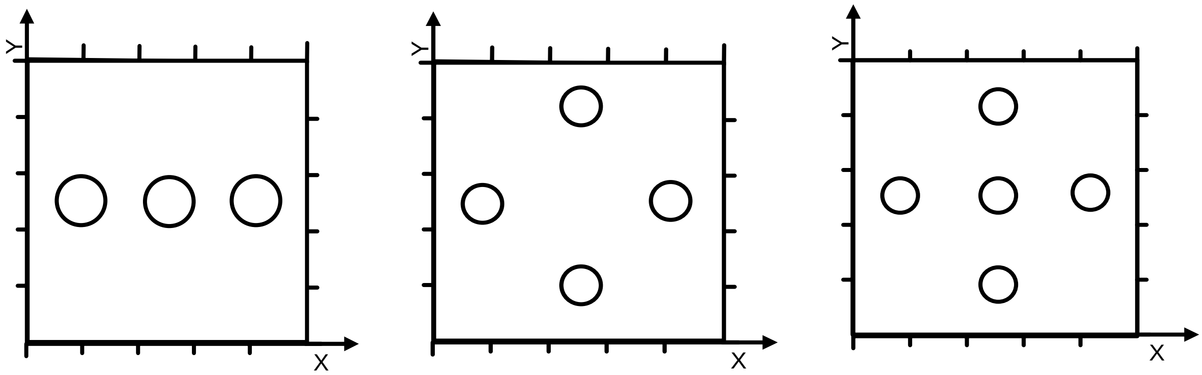
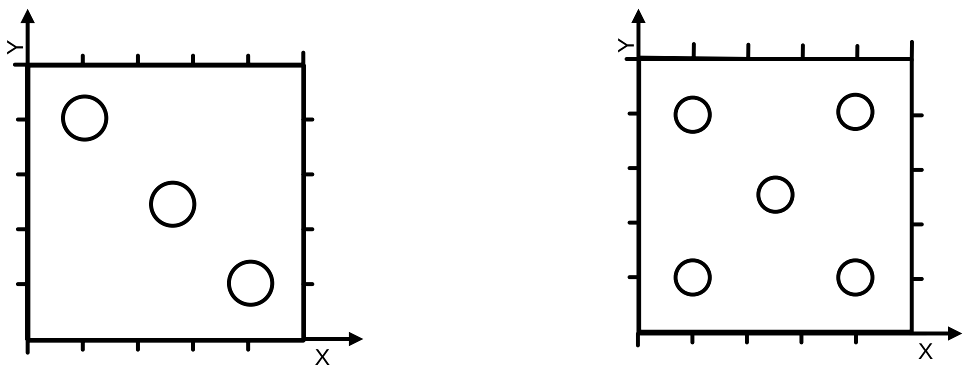


Figure 4: Circular cutout distributed equally at two positions.



(a) **(b)** **(c)**

Figure 5: Circular cutout distributed equally more than two portions into different positions.



(a) (b)

Figure6:Circular Cutout Distributed Equally At Different Orientations.

4. Findings

4.1 Case 1

The natural frequency of the plate with single cutout for different percentage of mass reduction is compared with the natural frequency of the bare plate for 11 modes are shown in Table 1 and Figure 7. In this case, mass reduction of the plate by single cutout is considered from 0 to 15% of the bare plate mass, where 0% indicates the

Table 1: Natural Frequency for Different Sizes of a Single Cutout at Middle

Mode No	Bare Plate Natural Frequency (Hz)	Natural Frequency (Hz) for 3% mass reduction	% of deviation from bare plate	Natural Frequency (Hz) for 6% mass reduction	% of deviation from bare plate	Natural Frequency (Hz) for **10%** mass reduction	%of deviation from bare plate	Natural Frequency(Hz) for **15%** mass reduction	% of deviation from bare plate
1st	27.306	27.127	-0.655	26.787	-1.900	26.963	-1.256	29.153	6.764
2nd	68.288	68.291	0.0043	67.609	-0.994	64.596	-5.406	59.975	**-12.173**
3rd	68.288	68.298	0.0146	67.619	-0.979	64.612	-5.383	59.958	**-12.198**
4th	109.25	108.85	-0.366	107.45	-1.647	104.84	-4.036	100.15	**-8.329**
5th	136.64	135.94	-0.512	133.81	-2.068	129.75	-5.042	122.89	**-10.062**
6th	136.64	135.65	-0.724	138.75	1.547	155.23	**13.60**	189.83	**38.927**
7th	177.59	177.62	0.0168	175.90	-0.947	171.82	-3.249	169.40	-4.608
8th	177.59	177.64	0.0281	175.94	-0.926	171.89	-3.209	169.35	-4.636
9th	232.42	232.03	-0.167	226.32	**-2.623**	220.87	-4.969	239.96	3.244
10th	232.42	231.98	-0.189	226.42	-2.580	220.89	-4.960	240.02	3.269
11th	245.9	245.69	-0.085	250.07	1.697	263.2	**7.035**	260.26	5.839

bare plate. If the reduction of mass is increased as shown in Figure 7, the natural frequency of the plate deviates from the natural frequency of the bare plateshows visible amount of deviation after 5% of the total mass reduction. For 10% mass reduction, the deviation of natural frequency compared to the bare plates shows within 1-13.6% where the maximum deviation occurs at 6th mode (Table1). Again for 15% mass reduction, the deviation of natural frequency compared to the bare plates shows within 3-38.9% where the maximum deviation occurs also at 6thmode(Table1).

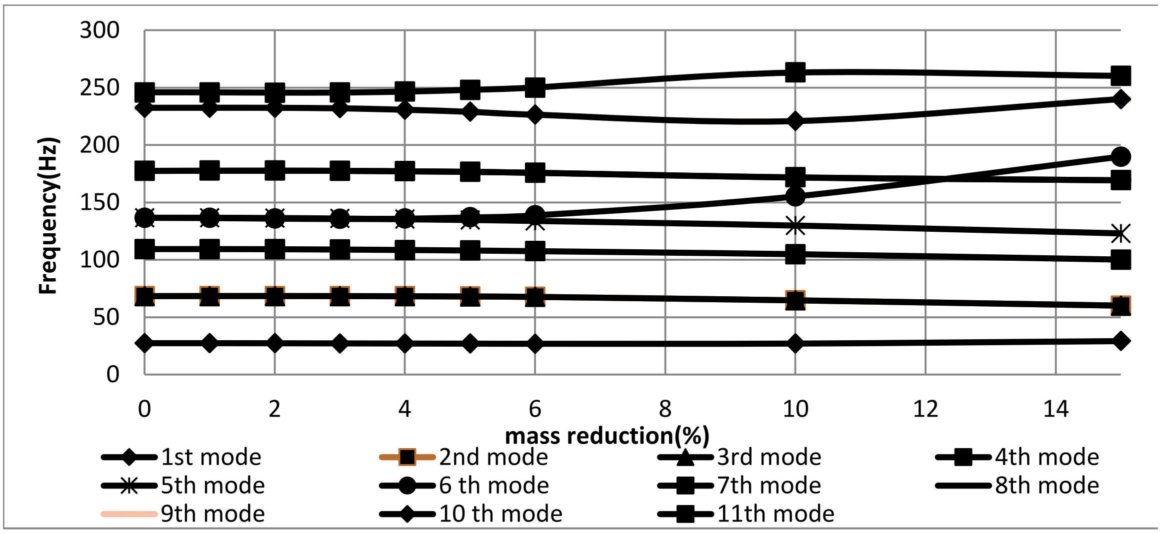


Figure 7:Natural Frequency vs. Mass Reduction for Plate with Single Cutout (case1)

4.2 Case 2

Cutout at position 1 is now changes to position 2 and position 3 (Figure 3) separately to observe the change of natural frequency compared to bare plate and case1. Mass reduction is considered in this case only for 10% & 15 % mass of total bare plate mass.For cutout with 10% reduction of the total mass, the deviation is between 1-13.6% when it is situated at position 1 (Table 2). The % of deviation shows between1-5.68 for position 2 which is much less than position 1(case 1). The maximum deviation which was at 6th mode in position 1 is now shifted to 8th mode. The % of deviation shows between1-8.1% for position 3 which is much less than position1 (case 1). The maximum deviation which was at 6th mode in position 1 is now shifted to 8th mode. The data shows for that the maximum deviation increases than position 2.

For cutout with 15% reduction, the deviation is between 3.24-38.92% when it is situated at position 1 (Table 3). The % of deviation shows between 1.2-23.25 forposition 2 which is much less than position 1(case 1). The maximum deviation which was at 6th mode in position 1 is now shifted to 8th mode (Table 3). The % of deviation shows between1-8.1 for position 3 which is much less than position 1(case 1). The maximum deviation which was at 6th mode in position 1 isnow shifted to 8th mode. The data shows for that the maximum deviation increases than position 2(Figure 9).

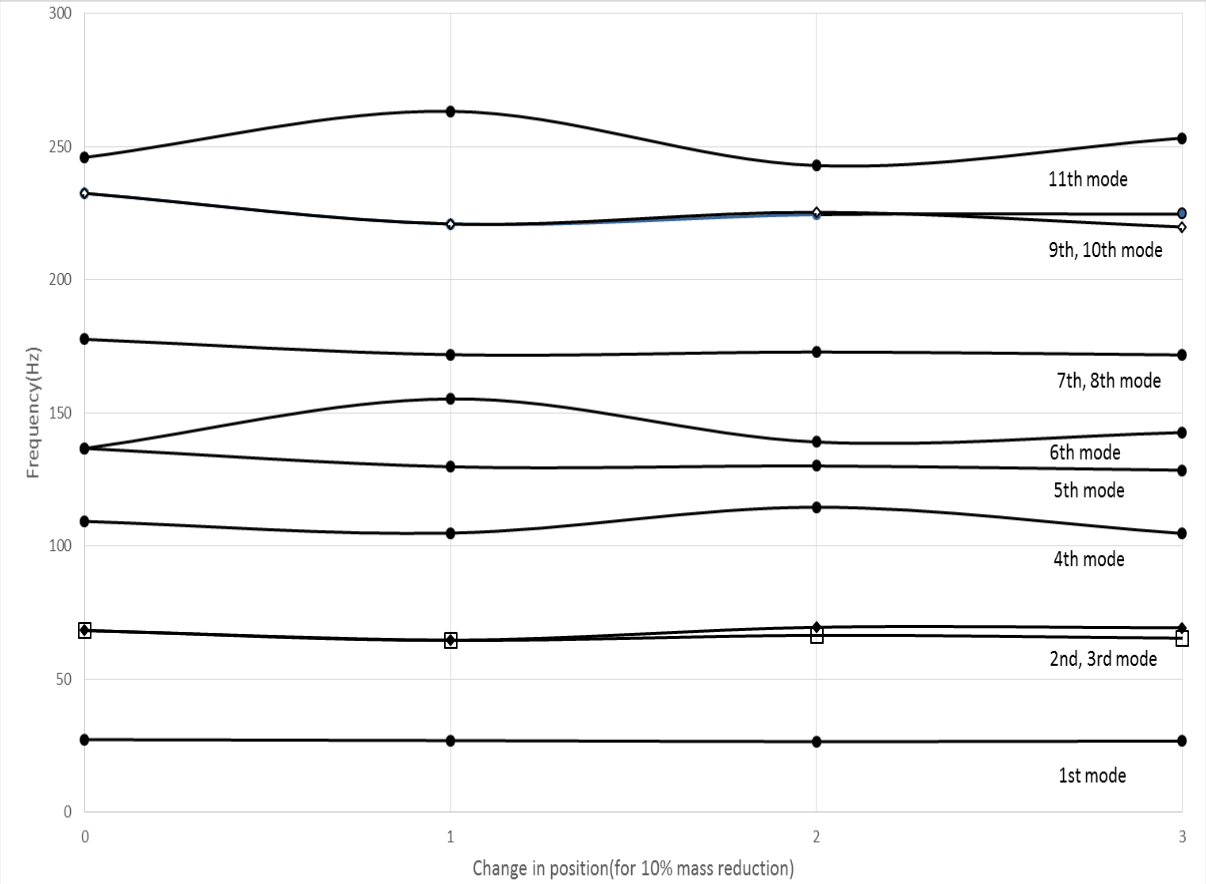


Figure 8: Natural frequency VS. Position of Single Cutout (1, 2 and 3 Indicates the Positions as Figure 3)

Table 2: Natural Frequency of 10% Mass Reduction Due to Change in Position

Mode No.	Bare plate Natural Frequency(Hz)	Natural Frequency(Hz) for cutout at Position 1	% of deviation from bare plate	Natural Frequency (Hz) for cutout at Position 2	% of deviation from bare plate	Natural Frequency(Hz) for cutout at Position 3	% of deviation from bare plate
1st	27.306	26.963	-1.2561	26.545	-2.7869	26.792	-1.8827
2nd	68.288	64.596	-5.4065	66.464	-2.6710	65.386	-4.2496
3rd	68.288	64.612	-5.3830	69.485	1.7528	69.271	1.4394
4th	109.25	104.84	-4.0366	114.56	4.8604	104.7	-4.1647
5th	136.64	129.75	-5.0424	130.05	-4.8228	128.46	-5.9865
6th	136.64	155.23	**13.6050**	139.03	1.7491	142.59	4.3545
7th	177.59	171.82	-3.2490	172.82	-2.6859	171.73	-3.2997
8th	177.59	171.89	-3.2096	187.68	**5.6816**	192.01	**8.1198**
9th	232.42	220.87	-4.9694	224.42	-3.4420	224.58	-3.3732
10th	232.42	220.89	-4.9608	225.31	-3.0591	219.75	-5.4513
11th	245.9	263.2	**7.0353**	242.82	-1.2525	253.16	2.9524

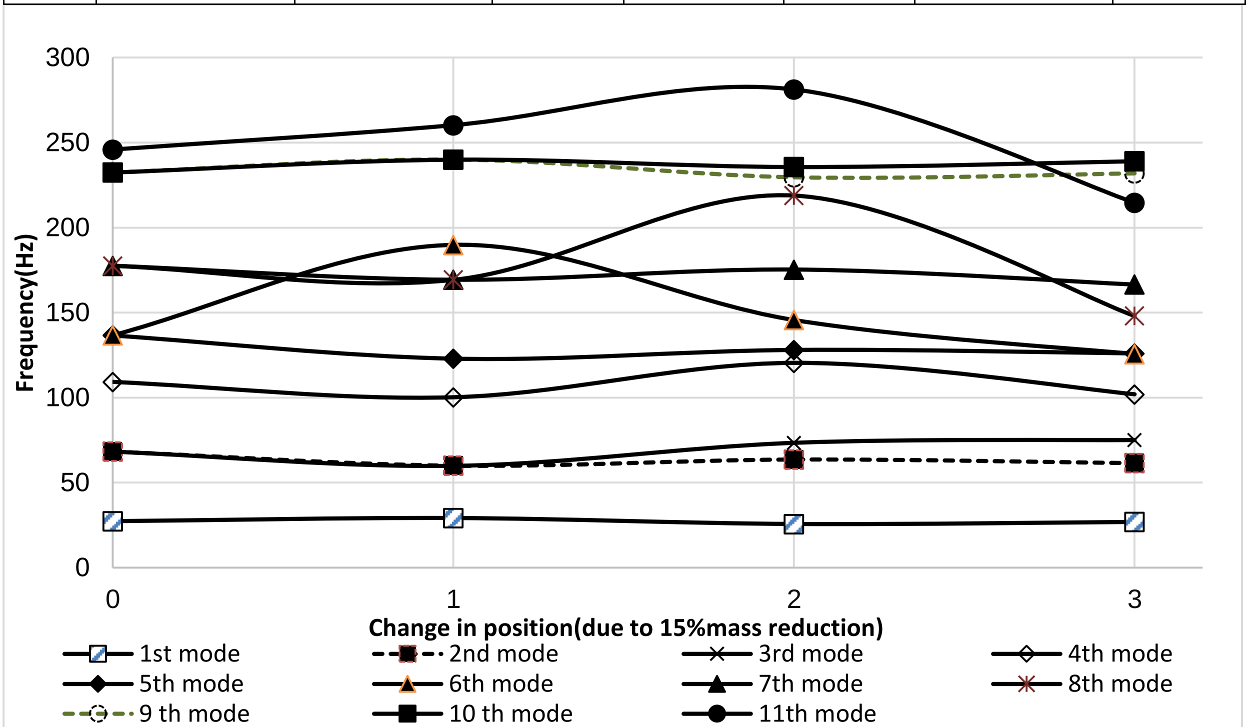


Figure 9: Natural Frequency vs. Position of Single Cutout

Table 3: Natural Frequency of 15% Mass Reduction Due to Change in Position

Mode No	Bare Plate Natural Frequency (Hz)	Natural Frequency (Hz) for cutout at Position 1	% of deviation from bare plate	Natural Frequency(Hz) for cutout at Position 2	% of deviation from bare plate	Natural Frequency(Hz) for cutout at Position 3	% of deviation from bare plate
1st	27.306	29.153	6.7640	25.656	-6.042	26.86	-1.633
2nd	68.288	59.975	-12.173	63.616	-6.841	61.459	-10.000
3rd	68.288	59.958	-12.198	73.452	7.562	75.085	9.953
4th	109.25	100.15	-8.329	120.53	10.325	101.92	-6.709
5th	136.64	122.89	-10.062	128.06	-6.2792	126	-7.786
6th	136.64	189.83	**38.927**	145.62	6.572	125.61	-8.072
7th	177.59	169.406	-4.608	175.35	-1.261	166.6	-6.188
8th	177.59	169.356	-4.636	218.89	**23.255**	148.15	**-16.577**
9th	232.42	239.96	3.244	229.62	-1.204	231.91	-0.219
10th	232.42	240.02	3.2699	235.67	1.398	239.11	2.878
11th	245.9	260.26	5.839	281.3	14.396	214.63	-12.716

4.3 Case 3

Cutout at position1 (case 1) is now divided into two cutout to observe the change of natural frequency compared to bare plate. Mass reduction is considered in this case only for 10% & 15 %, which show significant deviation of natural frequency from bare plate.

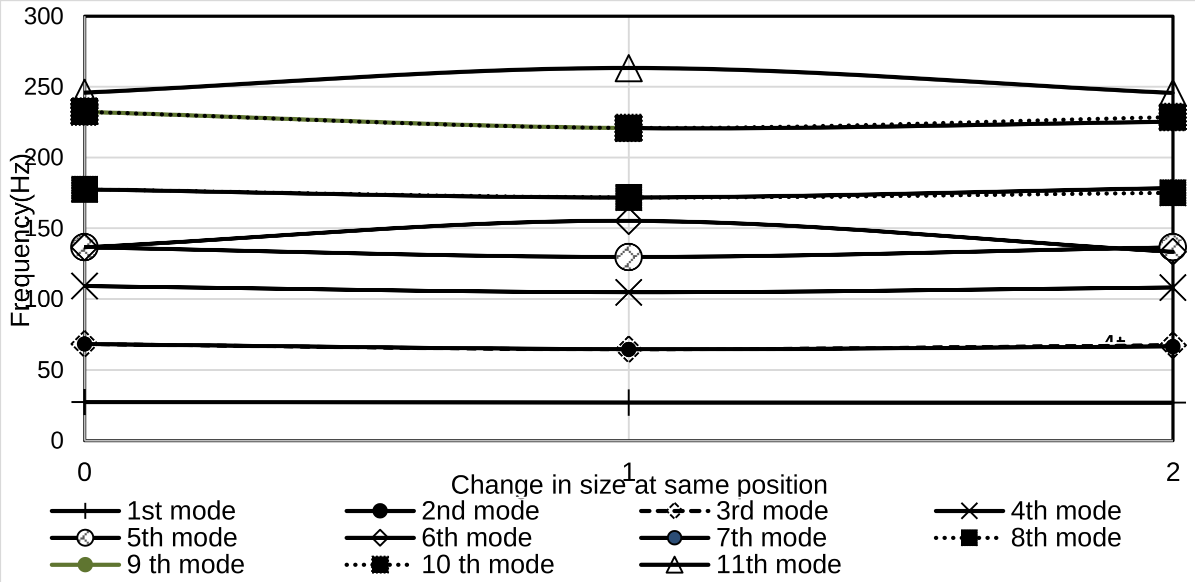


Figure 10: Natural Frequency VS number of cutout (for 10% mass reduction)

Table 4: Natural frequency Due to Distribution of cutout (10%)

Mode no.	Bare Plate Natural frequency (Hz)	Natural frequency (Hz) for single cutout (case 1)	% of deviation from bare plate	Natural frequency (Hz) for two cutout	% of deviation from bare plate
1st	27.306	26.963	-1.25613	26.866	-1.61137
2nd	68.288	64.596	-5.40651	66.611	-2.45578
3rd	68.288	64.612	-5.38308	67.467	-1.20226
4th	109.25	104.84	-4.03661	108.27	-0.89703
5th	136.64	129.75	-5.04245	136.66	0.014637
6th	136.64	155.23	**13.60509**	133.47	-2.31996
7th	177.59	171.82	-3.24906	178.55	0.540571
8th	177.59	171.89	-3.20964	175.1	-1.40211
9th	232.42	220.87	-4.96945	225.34	**-3.04621**
10th	232.42	220.89	-4.96085	228.63	-1.63067
11th	245.9	263.2	7.03538	245.69	-0.0854

For single cutout with 10% reduction, the deviation is between 1.2-13.6% when it is situated at position 1 (Table 4). The % of deviation shows between0.1-3.04 for 2 Cutout which is much less than single cutout (case 1). The maximum deviation which was at 6th mode at the middle is now shifted to 9th mode for two cutouts. In this case the data shows that the deviation is less than the shifting the position of the cutout.

For single cutout with 15% reduction, the deviation is between 3.2-38.9% when it is situated at position 1 (Table 5). The % of deviation shows between 0.13-8.71% for 2 cutouts which is much less than single cutout of case 1(Table 5). The maximum deviation which was at 6th mode at the middle is now shifted to 9th mode for two cutouts (Figure 11). In this case the data shows that the deviation is less than the shifting the position of the cutout.

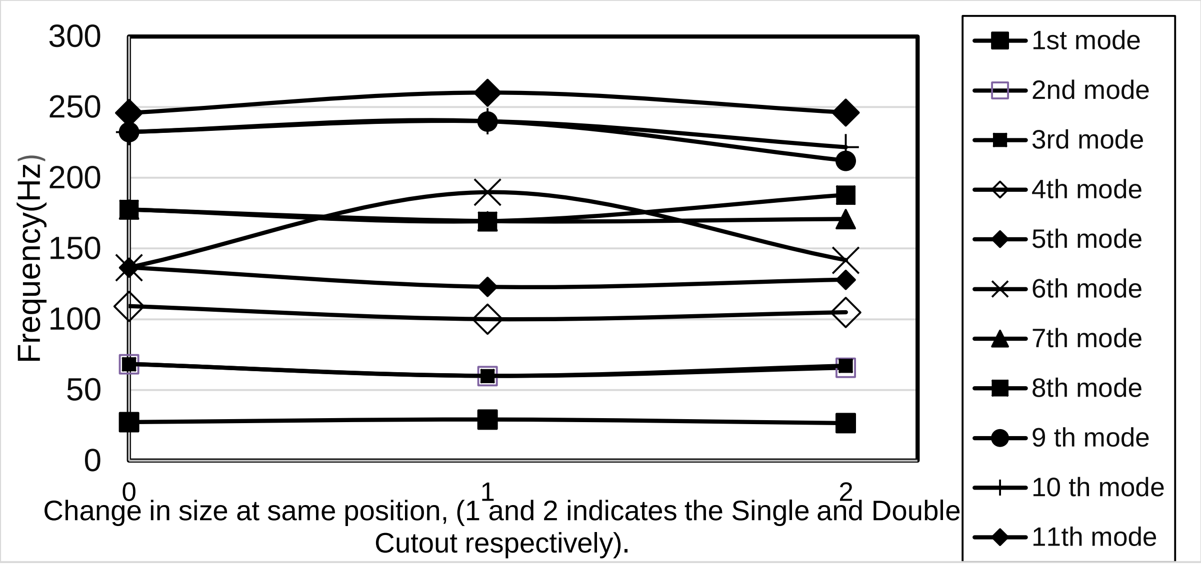


Figure 11: Natural Frequency vs Distribution of Mass Reduction (15% mass reduction)

Table 5: Natural Frequency Due To Distribution of Cutout (15%)

Mode no.	Bare Plate Natural Frequency(Hz)	Natural Frequency (Hz) for single cutout	% of deviation from bare plate	Natural Frequency(Hz) for double cutout	% of deviation from bare plate
1st	27.306	29.153	6.764081	26.567	-2.70636
2nd	68.288	59.975	-12.1734	65.663	-3.84401
3rd	68.288	59.958	-12.1983	67.052	-1.80998
4th	109.25	100.15	-8.32952	104.97	-3.91762
5th	136.64	122.89	-10.0629	128	-6.32319
6th	136.64	189.83	**38.92711**	141.83	3.798302
7th	177.59	169.406	-4.60837	170.87	-3.784
8th	177.59	169.356	-4.63652	187.87	5.788614
9th	232.42	239.96	3.244127	212.17	**-8.71268**
10th	232.42	240.02	3.269942	221.74	-4.59513
11th	245.9	260.26	5.839772	246.22	0.130134

4.4 Case 4

Cutout at position 1 (case 1) is now divided into 2, 3 and 4 circular cutouts to observe the change of natural frequency compared to bare plate. Mass reduction is considered in this case only for 10%.For single cutout the deviation is between 1.2-13.6% when it is situated at position 1 **(**Table 6).The % of deviation shows between0.1-3.04 for 2 cutouts which is much less than single cutout of case 1(Table 6).The % of deviation shows between0.03-1.2 for 4 cutouts which is much less than 3 cutouts. It shows that the more the no of cut the deviation becomes lower. The maximum deviation which was at 6th mode for single cutout is now shifted to 5th mode for four cutouts (Table 6).

Table 6: Natural Frequency due to Distribution of Cutout (10%)

Mode No	Bare Plate Natural Frequency(Hz)	Natural Frequency (Hz) for 1 cutout (case 1)	Natural Frequency(Hz)for 2 cutout	% of deviation from bare plate	Natural Frequency(Hz) for 3 cutout	% of deviation from bare plate	Natural Frequency (Hz) for 4 cutout	% of deviation from bare plate	Natural Frequency(Hz) for 5 cutout	% of deviation from bare plate
1st	27.3	26.96	26.86	1.61	26.85	1.66	27.6	-1.07	27.05	0.926
2nd	68.28	64.59	66.6	2.46	67.1	1.78	69.23	-1.388	67.74	0.789
3rd	68.29	64.61	67.46	1.202	67.96	0.47	69.01	-1.06	67.77	0.758
4th	109.2	104.8	108.2	0.897	108.5	0.68	111.6	-2.15	109.2	0.045
5th	136.6	129.7	136.6	-0.01	134.67	1.441	138.3	-1.22	134.8	**1.280**
6th	136.6	155.2	133.4	2.319	134.1	**1.836**	139.1	-1.81	135.3	0.966
7th	177.59	171.8	178.6	-0.54	176.3	0.755	180.3	-1.52	176.5	0.613
8th	177.59	171.8	175.1	1.402	176.5	0.591	180.7	-1.78	176.5	0.591
9th	232.4	220.8	225.3	**3.046**	229.8	1.118	236.9	-1.92	232.2	0.086
10th	232.4	220.9	228.6	1.630	230.5	0.817	238.35	**-2.55**	232.3	0.038
11th	245.9	263.2	245.6	0.085	244.9	0.378	251.03	-2.08	244.5	0.54

4.5 Case 5

Previously we saw, for three circular cutouts as shown in Figure 5(a), the maximum frequency deviation was 1.83% (6th mode). At the time we change the orientation of these cutouts as shown in Figure 6(a) the maximum frequency deviation reduces to 1.59%. So the frequency becomes closer to the natural frequency of the bare plate. But the maximum deviation changes its mode. Now it occurs at first mode.

Again when the orientation of five circular cutouts is changed as shown in Figure 6(b). We can see that the maximum frequency deviation does not change more. It just changes its mode. Previously the maximum deviation occurs at fifth mode now it shifts to fourth mode.

Table 7: Natural Frequency due to Orientation of cutout (10%)

Mode NO	Bare Plate Natural frequency (Hz)	Natural frequency for 3cutout(Hz)	% of deviation from bare plate	Natural frequency (Hz) for 5cutout	% of deviation from bare plate
1	27.306	26.871	-**1.59306**	27.095	-0.77272
2	68.288	67.33	-1.40288	67.832	-0.66776
3	68.288	67.993	-0.43199	67.882	-0.59454
4	109.25	107.57	-1.53776	107.85	-**1.28146**
5	136.64	134.99	-1.20755	136.13	-0.37324
6	136.64	134.67	-1.44174	135.16	-1.08314
7	177.59	176.16	-0.80523	176.72	-0.48989
8	177.59	176.89	-0.39417	176.76	-0.46737
9	232.42	230.22	-0.94656	232.28	-0.06024
10	232.42	230.78	-0.70562	232.31	-0.04733
11	245.9	244.51	-0.56527	244.69	-0.49207

5. Validation of Results

One circular cutout is considerd at position 1(caase 1) and the dimensionalized natural frequencies are converted into non-dimensionalized natural frequencies in order to compare the results with ICCM (Kwak& Han 2007) The comparisons are accounted up to 6th mode. The following Figure is showing the validated result for non-dimensionalized natural frequencies of the simply supported square plate with circular cutout. The non-dimensionalised natural frequency is compared to the ICCM. The result shows good agreement with the ICCM (Kwak& Han 2007).

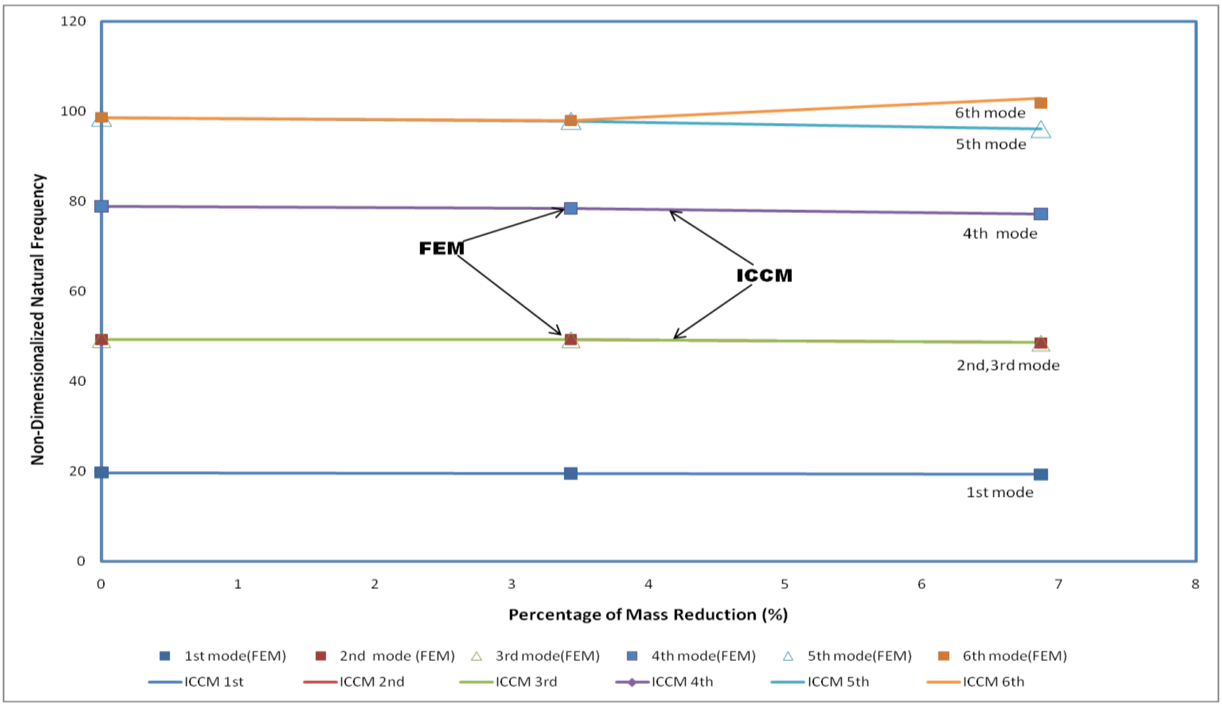


Figure 12: Non-Dimensionalised Natural Frequency vs Percentage of Mass Reduction

6. Conclusion

In this paper a comparative analysis of circular cutout of plate has been done. The investigation has been carried out for single cutout of different sizes and then for different position, distribution and orientation of that single cutout. The findings can be summarized as bellow:

1) Mass reduction by circular cutout affects the natural frequency of a plate mainly after a certain value and it is significant above 5% mass reduction of total mass of the bare plate.
2) Change of position and distribution of cutout affects the natural frequency of the plate. Maximum deviation of natural frequency occurs due to cutout at middle position of the plate and other positions also deviates the natural frequency from the bare plate.
3) Increases in the no of cutout with this specified orientation (Figure 5) decreases the deviation of the natural frequency.
4) Validation has been done for single cutout and the validated results are in good agreement.
5) Limitation:Different orientations of circular cutout change the natural frequency of the plate. In this literature we have analyzed only for two different orientations. Other orientations also show the significant change of natural frequencies.

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